II. *Semidiurnal Tide.*

Mean Lunitidal Interval (observed),

H. W.	L. W.
$-0^h\ 26^m{\cdot}7$.	$6^h\ 1^m{\cdot}1$.

III. "On Musical Duodenes, or the Theory of Constructing Instruments with Fixed Tones in Just or Practically Just Intonation." By ALEXANDER J. ELLIS, F.R.S., F.S.A., F.C.P.S., F.C.P. Received October 28, 1874.

This paper is intended to complete and supplement three papers on Music which I have already read before the Royal Society[1]. It contains a more complete theory of temperament, embracing that indicated by Helmholtz[2], but not worked out by him, and its application to the theory of constructing musical instruments with an intonation practically just, without change of fingering, and, if there are three or four performers, without change of mechanism. The name *Duodene* refers to that collection of *twelve* notes, suitable to the present manuals, which is made the unit of construction. To obtain its precise form, and determine the number and value of all such duodenes as it is necessary to tune, I have been obliged to indicate a theory of harmonic scales and modulation, which I believe to be entirely new, and which has of course other uses. The great extent of the subject obliges me to confine this part of my paper to a mere indication.

A. *Notation of Pitch.*

The letters C, D, E, F, G, A, B indicate both musical tones and the number of vibrations made by the prime or lowest partial tone of each in a second; so that, C being known,

$$D=\tfrac{9}{8}\,C,\ E=\tfrac{5}{4}\,C,\ F=\tfrac{4}{3}\,C,\ G=\tfrac{3}{2}\,C,\ A=\tfrac{5}{3}\,C,\ B=\tfrac{15}{8}\,C.$$

The marks ♯ ♭ † ‡ ¶ ╙ are used for fractional multipliers, having the following names and values:—

$$\text{sharp, } \sharp = \frac{135}{128};\quad \text{flat, } \flat = \frac{128}{135},$$

$$\text{high, } \dagger = \frac{81}{80};\quad \text{low, } \ddagger = \frac{80}{81},$$

$$\text{skhismic, } \P = \frac{32805}{32768};\quad \text{hyposkhismic, } ╙ = \frac{32768}{32805},$$

[1] "On the Conditions, Extent, and Realization of a Perfect Musical Scale on Instruments with Fixed Tones," read Jan. 21, 1864, printed at length in Proceedings, vol. xiii. p. 93; "On the Physical Constitution and Relations of Musical Chords," and, lastly, "On the Temperament of Musical Instruments with Fixed Tones," both read on June 16, 1864, and printed at length in the Proceedings, vol. xiii. p. 392 and p. 404.

[2] Tonempfindungen, 3rd ed. p. 495.

the first two being written and read *after*, and the last four being written and read *before*, the letter which marks the note. Thus:—

$$\dagger A\flat = \text{high A flat} = \frac{81}{80} \cdot \frac{5}{3} \cdot \frac{128}{135} \cdot C = \frac{4}{5} C,$$

$$\ddagger G\sharp = \text{low G sharp} = \frac{80}{81} \cdot \frac{3}{2} \cdot \frac{135}{128} \cdot C = \frac{25}{16} C.$$

The precise pitch of every tone is therefore indicated by its symbol, when C is known.

When *Italic* letters are used, $C_{,,}$ C, c, c', c'', &c. indicate Octaves, and C is the pitch of the lowest note of the violoncello. When Roman letters are used, no such relative values are attributed to large and small letters.

B. *Temperament.*

All intervals here considered can be made up of Fifths, major Thirds, and Octaves, taken up or down. In other words, the ratio of the vibration-numbers of any two tones can be represented by $\left(\frac{3}{2}\right)^m \cdot \left(\frac{5}{4}\right)^n \cdot 2^p$, where m, n, and p are zero, or some positive or negative integers. In Table I. the ratios of the vibrational numbers of all the tones in the same horizontal line is $\left(\frac{5}{4}\right)^n \cdot 2^p$, and in the same vertical column is $\left(\frac{3}{2}\right)^m \cdot 2^p$, the p or Octave being left indeterminate. If, then, we proceed from any note supposed to be the lower, first horizontally to the column containing a second note, supposed to be higher and in the same Octave, and then vertically to the second note itself, and multiply or divide by $\frac{5}{4}$ for each horizontal step, according as it is to the right or left, and multiply or divide by $\frac{3}{2}$ for each vertical step, according as it is upwards or downwards, and finally multiply or divide the result by 2, until the final result lies between 1 and 2, that fraction will be the ratio of the higher vibrational number to the lower. Thus C to F♯ gives C to E, $\frac{5}{4}$; E to B, $\frac{3}{2}$; B to F♯, $\frac{3}{2}$; and $\frac{5}{4} \cdot \left(\frac{3}{2}\right)^2 = \frac{45}{16}$, which, being greater than 2, on being divided by 2 gives the correct ratio of the *Tritone* C to F♯ as $\frac{45}{32}$. An inspection of the Table then shows that ‡D to ‡G♯, †G♭ to †C, &c. are the same intervals as C to F♯, because they have the same relative position, and this is sometimes very convenient.

But as 2, 3, 5 are primes, no one tone in Table I., supposing it to be indefinitely extended, will have the same pitch as any other tone. The object of Temperament generally is to obviate this inconvenience, by slightly altering the ratios of the Fifth or major Third, or both. From doing so, as Helmholtz has proved, more or less dissonance must result.

Another object, therefore, is to make that dissonance as little annoying as possible.

We find immediately by actual multiplication,

$$\left(\frac{3}{2}\right)^{12}\cdot\left(\frac{1}{2}\right)^{7}=\frac{531441}{524288}=\dagger\P,\text{ the comma of Pythagoras,} \quad (1)$$

$$\left(\frac{4}{5}\right)^{3}\cdot 2 = \frac{128}{125} = \dagger\dagger\|\!\flat,\text{ the diesis} \quad (2)$$

Multiplying these equations together, and extracting the cube root,

$$\left(\frac{3}{2}\right)^{4}\cdot\frac{4}{5}\cdot\left(\frac{1}{2}\right)^{2}=\frac{81}{80}=\dagger,\text{ the comma (of Didymus)} \quad (3)$$

Dividing (1) by (3),

$$\left(\frac{3}{2}\right)^{8}\cdot\frac{5}{4}\cdot\left(\frac{1}{2}\right)^{5}=\frac{32805}{32768}=\P,\text{ the skhisma.} \quad (4)$$

On these equations depend all *uniform* temperaments in which every Fifth and major Third preserves the same ratio throughout.

Let V, T, k, s be any four fractions having relations similar to $\frac{3}{2}$, $\frac{5}{4}$, † and ¶ respectively in (1) and (2); then

$$V^{12}\div 2^{7}=ks,\text{ and } 2\div T^{3}=k^{2}\div s. \quad (5, 6)$$

Subtracting the logarithms of (1) and (2) from the logarithms of (5) and (6) respectively, we find

$$\log V=\log\frac{3}{2}-\frac{1}{12}\cdot(\log\dagger-\log k+\log\P-\log s), \quad (7)$$

$$\log T=\log\frac{5}{4}+\frac{2}{3}\cdot(\log\dagger-\log k)-\frac{1}{3}(\log\P-\log s), \quad (8)$$

which are the fundamental equations of all temperament, and are identities, of course, for just intonation. As they contain 4 unknown expressions, two may be assumed and the rest found, giving rise to an endless variety of temperaments. Without discussing these generally, the following cases should be mentioned :—

Commatic System (for which $k=1$).—This is the only system discussed in my previous paper, where 50 cases were considered.

1. *Quintal* or Pythagorean Temperament. Assume $k=1$, and $V=\frac{3}{2}$, then by (7),

$$\log\dagger+\log\P=\log s;$$

whence by (8),

$$\log T=\log\frac{5}{4}+\log\dagger.$$

This temperament proves to be thoroughly unsuitable for harmony.

2. *Tertian*, Mesotonic or Mean Temperament. Assume $k=1$ and $T=\frac{5}{4}$, then by (8),

$$2\log\dagger=\log\P-\log s;$$

whence by (7),

$$\log V = \log \tfrac{3}{2} - \tfrac{1}{4} \log \dagger.$$

In my previous paper I have shown this to be the best suited for harmony of all in this system; but it requires 27 tones to the Octave.

3. *Hemitonic* or Equal Temperament. Assume $k=1$, and also $s=1$, then by (7),

$$\log V = \log \tfrac{3}{2} - \tfrac{1}{12} \cdot (\log \dagger + \log \P),$$

by (8),

$$\log T = \log \tfrac{5}{4} + \tfrac{1}{3} \cdot (2 \log \dagger - \log \P).$$

Since $\log \dagger = 0{\cdot}005\ 3950\ 319$ (whence $\frac{1}{11} \log \dagger = 0{\cdot}000\ 4904\ 574$) and $\log \P = 0{\cdot}000\ 4901\ 071$, we may, for all acoustical purposes, assume $\log \P = \frac{1}{11} \log \dagger$, and hence that

$$\log V = \log \tfrac{3}{2} - \tfrac{1}{11} \cdot \log \dagger = \log \tfrac{3}{2} - \log \P,$$

$$\log T = \log \tfrac{5}{4} + \tfrac{7}{11} \cdot \log \dagger = \log \tfrac{5}{4} + 7 \log \P.$$

This is the only uniform temperament which requires no more than 12 tones, and hence, although very ill suited to harmony, it has been theoretically adopted by almost all musicians.

Skhismatic System (for which $s=1$).

4. *Skhismic* or Arabic, according to Helmholtz's indication (*op. cit.* p. 441). Assume $s=1$ and $V=\frac{3}{2}$; then by (7),

$$\log k = \log \dagger + \log \P,$$

and by (8),

$$\log T = \log \tfrac{5}{4} - \log \P.$$

The Arabs tune 17 tones to form 16 perfect Fifths, and thus obtain 8 major Thirds which are only one skhisma too flat. But they do not use harmony in our sense of the word, although this would give almost just intonation in certain keys.

5. *Skhistic,* or Helmholtzian, as it may be called, because suggested by Helmholtz (*op. cit.* p. 495). Assume $s=1$ and $T=\frac{5}{4}$; then by (8),

$$2 \log k = 2 \log \dagger - \log \P,$$

and by (7),

$$\log V = \log \tfrac{3}{2} - \tfrac{1}{8} \log \P.$$

The name *skhistic* is derived from *skhist*, or by abbreviation σ, which is my name for $\sqrt[8]{\P}$. Since log $\dagger = 88$ log σ very nearly, one comma may be said to contain 88 skhists. Skhistic temperament is indistinguishable from just intonation, and I shall use it in the theory of constructing instruments. If in Table I. we suppose the horizontal intervals to be still $\frac{5}{4}$, but the vertical intervals to be $\frac{3}{2\sigma}$, and the sign † to stand for $\frac{81}{80}\sqrt{\P}$, while $\dagger\ddagger = 1$ as before, then this Table represents skhistic relations; so that if from any note, as E♭ (col. 5, line x), we proceed by 8 skhistic Fifths up to †B and then one major Third to the right, we find a tone D♯, which, when reduced to the same Octave, is identical with E♭. We thus find a number of *skhistic synonyms* shown in Tables II. & III.

We may express the *errors in the temperaments* just discussed in terms of skhists thus, using ♯56σ to mean "too sharp by 56 skhists," and so on, and 0 to mean "no error":—

Just.	1. Quintal.	2. Tertian.	3. Equal.	4. Skhismic.	5. Skhistic.
Minor Third	♭88σ	♭22σ	♭64σ	♯8σ	♭1σ
Major Third	♯88σ	0	♯56σ	♭8σ	0
Fourth	0	♯22σ	♯ 8σ	0	♯1σ
Fifth	0	♭22σ	♭ 8σ	0	♭1σ
Minor Sixth	♭88σ	0	♭56σ	♯8σ	0
Major Sixth	♯88σ	♯22σ	♯64σ	♭8σ	♯1σ

The error of one skhist is quite inappreciable by the most practised ears in melody, and can be detected harmonically only by very slow beats for tones in the highest Octaves used in music.

6. *Cyclic Temperaments.* The following method is far more general than that given in my previous paper (Proc. vol. xiii. p. 412). Put

$$m \,.\, \log V = v \,.\, \log 2, \qquad m \,.\, \log k = q \,.\, \log 2,$$
$$m \,.\, \log T = t \,.\, \log 2, \qquad m \,.\, \log s = z \,.\, \log 2,$$

and, after substituting these values for log V, log T, log k, log s in the logarithms of equations (5) and (6), divide out by log 2, and multiply up by m. Then

$$12v - 7m = q + z, \qquad m - 3t = 2q - z \quad \ldots\ldots\ldots \quad (9, 10)$$

Take any integral values for q and z, and find the integral values which satisfy one of these indeterminate equations for v, m, or t, m, and substitute in the other, taking the resulting integral values of t or v respectively. The five integral values determine a *cycle* in which the Octave is divided into m aliquot parts, which may be termed *octs*, v of which make a Fifth, t a major Third, q a comma, and z a skhisma of "the cycle of m." Most of the results are valueless, but the following present either theoretical convenience or historical interest:—

No.	Cycle of m.	Fifth, v.	Major Third, t.	Comma, q.	Skhisma, z.
1	30103	17609	9691	539	48
2	3010	1761	969	55	7
3	301	176	97	5	0
4	53	31	17	1	0
5	53	31	18	0	1
6	31	18	10	0	−1
7	12	7	4	0	0

Of these, the first three are here, I believe, for the first time shown to be true cyclic temperaments.

1. The cycle of 30103 is such an excellent representative of just intonation (giving even 6 octs for the skhist), that it can be used without sensible error, in place of ordinary logarithms, to reduce the relations of intervals to addition and subtraction, for general use among musicians or learners unacquainted with higher arithmetic. By dividing out by 100,000 we obtain almost precisely the five-figure logarithms of the intervals.

2. The cycle of 3010 is almost as correct, with smaller numbers.

3. The cycle of 301 is almost a perfect representation of skhistic temperament, in which the skhisma is eliminated, and for that reason becomes perhaps the most practical representation of general musical intonation.

4. The first cycle of 53 is Nicholas Mercator's representation of just intonation, but it is more correctly a representation of skhistic temperament, and not so good as No. 3[1].

5. The second cycle of 53 is a very accurate representation of Pythagorean intonation, and has actually been proposed for the violin by Drobisch.

6. The cycle of 31 is Huyghens's Cyclus Harmonicus, and closely represents the tertian or mean temperament.

7. The cycle of 12 is the ordinary equal temperament, and its principal convenience consists in the very small number of its octs, here called Semitones.

Unequal Temperaments, whether they consist of 12 selected tones from uniform temperaments, or of 12 tones turned intentionally false (see my former paper, Proc. vol. xiii. pp. 414–417, for their theory), are now abandoned. But the difficulty of tuning equal temperament by estimation of ear, or even by the monochord, and of retaining the intonation of the piano or organ unchanged for even an hour, makes all temperaments in actual use really unequal. The difficulty of original tuning by estimation

[1] When this paper was read, I mentioned that this was the cycle used by Mr. Bosanquet in his paper read before the Royal Society on 30th January, 1873.

of ear in the case of skhistic temperament, where the Fifths have to be flattened by an almost inaudible skhist, is so much enhanced as to be insuperable except by Scheibler's method[1]. Hence it is necessary to find a practical substitute. This I term

Unequally Just Intonation.—Suppose that the 48 tones marked off by a dotted line in Table I. have to be tuned in this substitute for skhistic intonation. Tune C to the fork. Take 4 just Fifths up (or Fourths down), C to G, G to D, D to †A, †A to †E, without beats; and three just Fifths down (or Fourths up), C to F, F to B♭, B♭ to E♭. Then tune C to E as a just major Third up, without beats, and from E proceed to its just Fifth B, verifying the result by determining that it is a just major Third above G, and so on to G♯ up and ‡G down. Then if from G♯ we proceeded to the just Fifth, D♯, the resulting tone would be exactly one skhisma sharper than E♭, whereas in skhistic intonation it would be identical with E♭, as already shown. It is needless to say that no tuner could effect this exact difference of a skhisma, but he will come practically near it, and the error is that of the Fifth (the least of the errors) in equal temperament. If we were to proceed in this way for all the six columns of 8 tones marked off in Table I., we should have just major Thirds throughout, and just Fifths also in all but 5 cases—namely, ‡D♯♯ to ‡B, B♯ to ‡G, G♯ to E♭, †E to C♭, and †C to †A♭♭, each of which would be too flat by one skhisma. Since ‡G (Table I. col. 6, line *x*) is a just major Third below ‡B, and a just minor Third above ‡E, but ‡E is a whole skhisma flatter than ‡D♯♯ (col. 8, line *p*), which would be played for ‡E, it follows that the minor Third, ‡D♯♯ to ‡G, would be a skhisma too flat or close; and similarly that the minor Thirds, B♯ to E♭, G♯ to C♭, and †E to †A♭♭, or 4 minor Thirds on the whole, would be a skhisma too flat[2]. Hence this style of tuning gives 5 Fifths and 4 minor Thirds, as

[1] Calculate the logarithms of the ratios of all the skhistic tones by perpetual addition or subtraction of 0·1760300 (=log $\frac{3}{2}$−log σ) to or from 0, continually adding or subtracting 0·3010300 (=log 2) to make the results positive and lie between this and 0. Add the logarithm of the vibrational number of C, and then find the numbers (to three places of decimals) corresponding to these logarithms. This gives the vibrational numbers of all the skhistic tones in the Octave, of which 48 will be required. Subtract 4 from each of these values, and procure tuning-forks giving exactly the tones thus determined to at least the hundredth of a vibration in a second. These may be obtained of the great manufacturer of acoustic apparatus, Mons. R. Koenig, of Paris; but it is necessary to state that the English and German (not the French) system of counting vibrations is to be used. Then tune each tone roughly to the corresponding fork, and afterwards *sharpen* it until it beats 4 times in a second with the fork. By this means, and by this means only, with great care and attention, the pitch may probably be obtained with sufficient accuracy to distinguish skhistic from just intonation. And similarly for equal and tertian intonation.

[2] Taking the cycle of 30103, ‡G contains 17070, ‡B 26761, and ‡D♯♯ 9200 octs. Hence the Fifth, ‡D♯♯ to ‡B, has only 17561 in place of 17609 octs, and the minor Third, ‡D♯♯ to ‡G, only 7870, in place of 7918 octs—that is, in each case 48 octs too little; that is, these intervals are one skhisma too flat.

bad as the *best* intervals (Fifths) in equal temperament, and all the other intervals absolutely just. Hence the name *unequally just*. In future I shall consider that skhistic intonation is practically realized by unequally just intonation, for which the practical rule is :—*tune six tones making just major Thirds without beats* (as †F♭ to †A♭ to C to E to ‡G♯ to ‡B♯, line *t* in Table I.), *and from each of them tune seven other tones making just Fifths* (as C to G to D to †A to †E, and C to F to B♭ to E♭, in col. 5 of Table I.).

Saunders's "Tilting Action."—Before proceeding to show that 48 skhistic tones suffice for modern modulational music, it will be useful for future constructions to remark that a method of realizing all the effects which I contemplated by my duplex finger-board (Proc. vol. xiii. p. 422) has been invented by Mr. T. W. Saunders[1], by means of stops, which allow the manual and fingering to remain unaltered. Two sets of harmonium vibrators are arranged one behind the other, tuned in tertian intonation (by means of beats, counted by a pendulum, which gives a fairly accurate means of approximating to the correct result) as follows, the capital letters referring to the white or long digitals, and the small letters to the black or short digitals, a mode of distinction which I shall constantly employ :—

Back	B♯	d♭	C♯♯	e♭	F♭	E♯	g♭	F♯♯	a♭	G♯♯	b♭	C♭
Front	C	c♯	D	d♯	E	F	f♯	G	g♯	A	a♯	B

There are 12 stops[2], one corresponding to each digital in the Octave, which, by a "tilting action," enables one, and one only, out of the two vibrators in the same column, as shown above, to be "damped" at pleasure throughout all the Octaves of the instrument. When all the stops are pushed in, the front vibrators only are free, and any one may be exchanged for a back vibrator by pulling out its stop. Hence 24 out of 27 tones are under the command of the player; B♭♭, E♭♭, A♭♭ are omitted.

C. *Harmonic Scales.*

A series of tones, each of which is consonant with two other tones in the same series that are themselves consonant with each other, forms what I here mean by an *harmonic scale*. This was not the principle on which scales were originally formed; but this is the way in which the pitch of the tones must be determined for the just intonation of modern harmony and modulation.

[1] As Mr. Saunders has not patented his invention, I am unable to give more than the indications in the text, and refer to him personally, at E. Lachenal's Concertina Manufactory, 4 Little James Street, Bedford Row, W.C. His invention offers great facilities for the construction of experimental instruments in any uniform or just intonation. His harmonium was shown when this paper was read.

[2] In the specimen shown there are only 9 stops, the C♯♯, F♯♯, G♯♯ having been omitted; but as the principle admits of the construction of 12 stops as easily as 9, the complete form is mentioned in the text.

The Harmonic Elements are the Fifth, C × G $\left(=\frac{3}{2}\right)$, the major Third, C + E $\left(=\frac{5}{4}\right)$, and the minor Third, C − †E♭ $\left(=\frac{6}{5}\right)$, using a notation which I have found practically very convenient for representing the intervals between two tones; the symbols ×, +, − are not to be employed with any other meaning between the names of tones. In these elements it is supposed that either note may be raised or depressed by any number of Octaves, or be accompanied by such Octaves of itself.

†E♭	G	
	C	E

The Harmonic Cell, or Unit of Concord, consists of a major triad, C + E − G, and a minor triad, C − †E♭ + G, arranged as in the margin, and having the same First C, and hence the same Fifth G. The Fifth C × G is placed vertically, the two major thirds, C + E and †E♭ + G, are horizontal, and the two minor Thirds, C − †E♭ and E − G, slope obliquely from the bottom upwards to the left. These positions, then, replace the symbols ×, +, −. Allowing any one of the tones to be altered by any number of Octaves, or to have Octaves of itself added, and any tone to be taken as the First, this *cell,* whence all harmony is *developed,* contains every chord recognized by musicians as a concord in Tertian Harmony—that is, harmony depending on Octaves, Fifths, and *Thirds* alone, excluding natural Sevenths $\left(=\frac{7}{4}\right)$, which form Septimal Harmony. By Table I. cells can be readily constructed on any tone as a First.

†E♭	G	
†A♭	C	E
	F	A

The Harmonic Heptad, or Unit of Chord-relationship, consists of *two* cells, the First of one being the Fifth of the other, as in the margin. Allowing Octave variations as before, this contains all the three major and three minor triads which have C as one of their constituents, and are thus related in the first degree. Two of these chords, the minor triad A − C + E on the right or major side, and the major triad †A♭ + C − †E♭ on the left or minor side, connecting the two cells and due to their union, may be called *union triads,* to distinguish them from the four *cell triads.* The heptad also contains all *con-dissonant triads* (as I term them), consisting of three tones, two of which are consonant with C but dissonant with each other. Of these the *trine* †A♭ + C + E, which forms the central horizontal line, is most important for future work.

†B♭	D	
†E♭	G	B
†A♭	C	E
	F	A

The Harmonic Decad, or Unit of Harmony, consists of two heptads having a common cell, and hence of three cells, the Fifth of the first, lowest, or *subdominant* cell, and the Fifth of the second, middle, or *tonic* cell, being the First of the second cell, and First of the third, highest, or *dominant* cell, respectively. The decad contains three major and three minor cell triads, and two major and two minor union triads —that is, ten triads in all, together with all the discords possible without modulation. The First of the tonic cell is called the *tonic of the decad*, and gives its name to it. The example, therefore, is a C decad.

Harmonic Trichordals consist of three triads, one from each cell in a decad, and form eight groups. Contracting major triad and minor triad into *ma* and *mi* respectively (with Italian vowels), and naming the three-cell triads in order from bottom to top, these 8 trichordals are distinguished as follows in the C decad. The triads are spread out and marked by + and −, and the terminal triads are repeated in part with an interposed | to indicate the dissonant interval of a Pythagorean minor Third $=\frac{80}{81}\cdot\frac{6}{5}=\frac{32}{27}$, so that all the harmonies, consonant and dissonant, peculiar to any trichordal, may be collected at a glance.

i. *Mamama*.........	B −D	F+ A −C+ E −G+ B −D	F+ A
ii. *Mimama*.........	B −D	F−†Ab+C+ E −G+ B −D	F−†Ab
iii. *Mamima*.........	B −D	F+ A −C−†Eb+G+ B −D	F+ A
iv. *Mimima*.........	B −D	F−†Ab+C−†Eb+G+ B −D	F−†Ab
v. *Mamami*.........	†Bb+D	F+ A −C+ E −G−†Bb+D	F+ A
vi. *Mimami*.........	†Bb+D	F−†Ab+C+ E −G−†Bb+D	F−†Ab
vii. *Mamimi*.........	†Bb+D	F+ A −C−†Eb+G−†Bb+D	F+ A
viii. *Mimimi*.........	†Bb+D	F−†Ab+C−†Eb+G−†Bb+D	F−†Ab

Each of these 8 trichordals contains 7 tones, and when these are reduced to one Octave and sounded in order of pitch, they form that particular scale in which a piece of music is usually written. But in repeating them each may begin on any tone of the seven, giving 7 *modes* (in the ancient Greek sense) to each trichordal. To distinguish these, change the *m* of the name of the triad containing the initial tone into *p* when it is its First (*p* rima), *t* when it is its Third (*t* ertia), and *qu* when it is its Fifth (*qu* inta), which last is of course required for the highest or dominant triad only. The final cadence fully distinguishes the 56 resulting *harmonic scales*. Of these I append such as are usually acknowledged, making them all begin with C, and changing the decad accordingly. Between the tones I use (.) for the Semitone $\frac{16}{15}$, (:) for the high Semitone $\frac{27}{25}$, (..) for the minor Tone $\frac{10}{9}$, (...) for the major Tone $\frac{9}{8}$, and (.·.) for the augmented tone $\frac{75}{64}$.

1. C *mapáma*, or ordinary scale of C *major*.

   c ... *d* .. *e* . *f* ... *g* .. *a* ... *b* . *c'*.

2. F *mamapá*, one of Helmholtz's modes of the Fourth, or *Quartengeschlecht*.

   c .. ‡*d* ... *e* . *f* ... *g* .. *a* . *b*♭ ... *c'*.

3. C *mipáma*, Helmholtz's minor-major mode, or *Moll-Durgeschlecht*.

   c ... *d* .. *e* . *f* ... *g* . †*a*♭ .·. *b* . *c'*.

4. C *mapíma*, Helmholtz's mode of the minor Seventh with the leading note, or *Septimengeschlecht mit dem Leittone*, a very usual form of the modern ascending scale of C minor.

   c . . . *d* . †*e*♭ . . *f* . . . *g* . . *a* . . . *b* . *c*′.

5. C *mipíma*, the theoretical modern ascending scale of C minor.

   c . . . *d* . †*e*♭ . . *f* . . . *g* . †*a*♭ ∴ *b* . *c*′.

6. F *mamapí*, considered by Helmholtz (*op. cit.* p. 434. no. 6) as a variant of the mode of the minor Seventh.

   c . . ‡*d* : †*e*♭ . . *f* . . . *g* . . *a* . *b*♭ . . . *c*′.

7. C *mapími*, Helmholtz's mode of the minor Seventh without the leading note.

   c . . . *d* . †*e*♭ . . *f* . . . *g* . . *a* : †*b*♭ . . *c*′.

8. C *mipími*, Helmholtz's mode of the minor Third, or *Terzengeschlecht*, the ordinary form of the modern descending scale of C minor.

   c . . . *d* . †*e*♭ . . *f* . . . *g* . †*a*♭ . . . †*b*♭ . . *c*′.

9. F *mimipí*, Helmholtz's mode of the minor Sixth, or *Sextengeschlecht.*

   c . *d*♭ . . . †*e*♭ . . *f* . . . *g* . †*a*♭ . . *b*♭ . . . *c*′.

These 56 harmonic scales are all that can be produced without modulation.

To retain old names as much as possible, C *mapáma* will be called C *major*, and all three, C *mipími, mipíma, mapíma,* will be considered as making up C *minor*, whilst other forms will be termed *unusual minor scales.* All these, however, and more of the 56 scales mentioned above, actually occur in modern music, at least for short phrases, although the usual *major* and *minor* alone characterize whole compositions.

D. *Modulation and Duodenation.*

Although a decad consists of complete triads and cells, yet it is evident that one or two of the cells may be made parts of other decads, and that the *union* triads may be regarded as parts of cells left incomplete. The tones forming these cells and unions are therefore *ambiguous*, and there is always a tendency to complete them in a different way from that in the original decad, or, in other words, to proceed to the other decads of which they form a part. By an extension of the term modulation, which originally referred to a mere change of mode, this change of decad might still be called *modulation*, although *decadation* might be more appropriate.

The Harmonic Heptadecad, or Unit of Modulation (or Decadation),

†d♭	†f	†a				*ru*	*fo*	*le*		
†g♭	†B♭	D	f♯			*su*	*To*	*Re*	*fi*	
†c♭	†E♭	G	B	‡d♯	or	*du*	*Mo*	*So*	*Ti*	*ri*
†f♭	†A♭	C	E	‡g♯		*fu*	*Lo*	*Do*	*Mi*	*se*
	d♭	F	A	‡c♯			*ro*	*Fa*	*La*	*de*
		b♭	‡d	‡f♯				*ta*	*ra*	*fe*

consists of seven interwoven decads, which are constructed on the seven tones of a heptad as tonics, and contains 24 tones. On the left is the heptadecad of C, in which the decad of C is printed in capitals, and the added tones necessary to complete the heptadecad in small letters. On the right *solfeggio* names are proposed as substitutes, to be pronounced with Italian vowels. These names are founded on those used by the Tonic Solfaists, and are suitable to *any* original tonic *Do*; and they are introduced because singers in just intonation should become accustomed to the "mental effect" of each of these tones in relation to the *Do* selected. The decads are named from the names of the tones in the original decad. The G decad is the dominant or *So* decad; the F the subdominant or *Fa* decad; the A the right relative, or major Sixth, or *La* decad; the †E♭ is the left relative or minor Third or *Mo* decad; the E is the right correlative or major Third, or *Mi* decad; and the †A♭ is the left correlative or minor Sixth or *Lo* decad—all with reference to the C or tonic or *Do* decad. These six decads are related to the original decad in the *first* degree. The dominant and subdominant decads have each *seven* tones, the relative and correlative decads have each *six* tones, in common with the original decad. The dominant decad raises two tones, F and A, by a comma, $\frac{81}{80}$, to †f and †a, and one, F, by a sharp, $\frac{135}{128}$, to f♯. The subdominant depresses two tones, †B♭ and D, by a comma, to b♭ and ‡d, and one, D, by a sharp, to d♭. These two decads are therefore equally related to the original. The right relative decad depresses one tone, D, by a comma, to ‡d, and raises three tones, F, C, G, by a low sharp, $\frac{25}{24}$, to ‡f♯, ‡c♯, ‡g♯. The right correlative decad raises one tone, F, by a sharp, to f♯, and three tones, C, G, D, by a low sharp, to ‡c♯, ‡g♯, ‡d♯. Hence the right relative is more nearly related to the original than the right correlative. Similarly the left relative, changing F in †f, and C, G, D into †c♭, †g♭, †d♭, is more nearly related than the left correlative, which changes D into d♭, and F, C, G into †f♭, †c♭, †g♭. In commatic temperaments, where ‡d, †f are not distinguished from D, F, the relative decads seem, like the just dominant and subdominant, to have 7 tones in common with the original, and similarly the dominant and subdominant decads appear to have 9 tones in common with the original. Hence various important confusions have arisen, of which it must suffice to have indicated the source. Since the most natural and easy harmonic trichordal, with the fullest and best harmonies, is undoubtedly the *mamama* or *major*, consisting of the central and right columns of the decad containing it, modulation (or decadation) to the *right* is more common than modulation to the *left*; and, owing to the closer relationship, modulation to the right relative is more common than into the right correlative, which generally occurs as a *vertical* (dominant) modulation from the latter. Vertical (dominant or subdominant) modulations are, however, the most common of all, unconsciously (owing to commatic temperament) into the subdominant (when the minor chord, ‡d—f+a, is used for the chord of the added Sixth, f+a | d), and consciously into the dominant (in which, however, only f♯, and not †a, is commonly recognized).

The vertical modulation is so common that it influences scales, producing actual tetrachordals, which are disguised in melody by being occasionally deprived of their extreme tones, so as to reduce their apparent number at any time to 7. The fourth chord may be added on to the name by a hyphen. Thus we have

C *ma-mapáma*...... b♭+‡d—f+a—c+e—g+b—d,

in which the b♭ is seldom touched except in the chord of the dominant Seventh, c+e—g | b♭, and then not in melody, but ‡d often comes into melody. Similarly we have

C *mapáma-ma*...... f+a—c+e—g+b—d+f♯—†a,

where †a is not touched in the melody. But in minor scales this is more marked, as

A *mimipá-ma*...... ‡d—f+a—c+e+‡g♯—b+‡d♯—f♯,

where ‡d and f♯ are not touched in the melody; so that the scale reads

e . *f* ∴ ‡*g*♯ . *a* ... *b* . *c'* ∴ ‡*d'*♯ . *e'*,

with 4 semitones and 2 augmented tones, which has an extremely strange effect[1]. Another scale of this kind is

A *mimipí-ma*...... ‡d—f+a—c+e—g+b+‡d♯—f♯,

which occurs in the modern treatment of Helmholtz's mode of the minor Sixth (No. 9, above). The apparent scale is

e . *f* ... *g* .. *a* ... *b* . *c* ∴ ‡*d'*♯ . *e'*,

which has 3 Semitones. These are, in fact, all cases of vertical modulation (or decadation); and it is only by recognizing this fact that we are able to reduce them to just intonation. They have not been, however, hitherto so conceived, and hence it became necessary, for the purposes of

[1] This scale and its harmonies are taken from C. Child Spenser's 'Rudimentary and Practical Treatise on Music,' vol. ii. p. 42. He does not acknowledge either ‡d or f♯; but he really uses ‡d in his second chord, ‡*d a* ‡*d' f'*, and he only avoids f♯ by using f+a... b+‡d♯ for the usual chord of the dominant Seventh, b+‡d♯—f♯ | a.

this paper, to explain them. A means of putting the strange-looking chords[1], f+a ... b+‡d♯, and f+a—c with ‡d♯, or f+a with ‡d♯, which they contain, under the hands of the performer on justly intoned instruments, is absolutely necessary.

The Harmonic Duodene or Element of Modulation, as distinguished from the heptadecad or unit of modulation, contains the 12 tones inclosed within an oblong in the figure of the heptadecad. It is seen to contain a complete decad of C and two additional tones, f♯ and d♭, which I term *mutators*, as each of them is part of two cells, and hence lead the old decad to *change* into the new decads containing them. Thus f♯ is part of

the vertical cell	†f †a
	    D f♯
		and of the lateral cell	D f♯
			    B ‡d♯,

and hence leads both to the dominant decad and to the right correlative decad. Again, d♭ is a part of

the vertical cell	d♭ F
	    b♭ ‡d
		and of the lateral cell	†f♭ †A♭
			    d♭ F,

and hence leads to the subdominant decad and to the left correlative decad. But these mutators, f♯ and d♭, also complete two scales left incomplete in the decad because they required vertical modulation (or decadation), namely,

†A♭ *mapáma*.... d♭+f —†a♭+c —†e♭+g—†b♭, and
E *mipími* .... a —c+ e —g+ b —d+ f♯,

and also complete the peculiar chords, d♭+F ... G+B, †A♭+C ... D+f♯, D♭+F—†A♭ with B, †A♭+C—†E♭ with f♯, which occur in the tetrachordals of minor scales already mentioned.

A *duodene*, then, consists of 12 tones, forming four *trines* of major Thirds arranged in three *quaternions* of Fifths. Hence the duodene constructed on the second tone C of any trine, †A♭+C+E, contains the *mapáma* or major of the first tone †A♭, the complete *decad* of the second tone C, and the *mipími* or common *descending* minor of the third tone E. It has therefore *three* tonics, †A♭, C, and E; but the tonic of the decad being most characteristic, this is called the *root* of the duodene, and the duodene is named after it.

Any duodene is clearly and sharply separated from its adjacent trines and quaternions, as shown in Table I., where the small innermost oblong marks off the duodene of C. For in the duodene the smallest intervals between two adjacent tones are the Semitone, $\frac{16}{15}$, (f♯ to G, B to C, E to F,

[1] The justification of these chords is that the interval *f* to ‡*d′*♯ $= 2 \cdot \frac{81}{80} \cdot \frac{9}{8} \cdot \frac{135}{128} \div \frac{4}{3} = \frac{7}{4} \cdot \frac{225}{224}$, and is hence very nearly the interval of the natural Seventh $= \frac{7}{4}$.

D to †E♭, G to †A♭, C to d♭); the Sharp, $\frac{135}{128}$, (F to f♯, d♭ to D); and the low Sharp, $\frac{25}{24}$, (†B♭ to B, †E♭ to E, †A♭ to A). But if we take the trine above, †f+†a+c♯, we have two intervals of a comma, †, (F to †f, A to †a), and one of a diaskhisma, †‖♭, (c♯ to d♭). If we take the trine below, g♭+b♭+‡d, we have the same intervals of a comma (b♭ to †B♭, ‡d to D), and a diaskhisma (f♯ to g♭). If we take the quaternion to the right, as ‡c♯ × ‡g♯ × ‡d♯ × a♯, we have three intervals of a diesis, ††‖♭, (a♯ to †B♭, ‡d♯ to †E♭, ‡g♯ to †A♭, and ‡c♯ to d♭); and similarly if we proceed to the left. Hence the intervals introduced by adjacent trines and quaternions are all less than two commas. In equal temperament no new intervals would be thus introduced; for all the Fifths are there so altered that the new upper trine, tempered †f+†a+c♯, would become *identical* with the original bottom trine, tempered d♭+F+A, except in order of terms; and the new quaternion to the right, tempered ‡c♯ × ‡g♯ × ‡d♯ × a♯, would be *identical* both in value and order of terms with the old quaternion to the left, tempered d♭ × †A♭ × †E♭ × †B♭. The consequence is that *only one duodene* exists for equal temperament, and the real nature of modulation is thoroughly disguised. In tertian temperament this would not be the case; the quaternions would be distinguished, but the trines would partly coincide, and hence some, but not all, of the meaning of modulation would be lost[1].

[1] If in Table I. the signs † ‡ be omitted, and the letters and the signs ♯ ♭ be taken to have their values in Tertian or any uniform commatic temperament (except the Equal, which is also skhismatic), the Table will represent the corresponding duodenes. But if the letters and signs ♯ ♭ are taken to have their value in the Equal temperament, so that

	C	D	E	F	G	A	B
	=D♭♭	E♭♭	F♭	G♭♭	A♭♭	B♭♭	C♭
and	=B♯	C♯♯	D♯♯	E♯	F♯♯	G♯♯	A♯♯

and

C♯	D♯	E♯	F♯	G♯	A♯	B♯
=D♭	E♭	F	G♭	A♭	B♭	C

(showing the utterly absurd relations between symbolization and signification), then the same Table will reduce to the one central duodene with its tones differently distributed. This will be still better shown by using

C cd D de E F fg G ga A ab B

for the 12 digitals on a piano, so that the central duodene and its adjacent trines and quaternions reduce to

cd	F	A	cd	F
fg	ab	D	fg	ab
B	de	G	B	de
E	ga	C	E	ga
A	cd	F	A	cd
D	fg	ab	D	fg

In skhistic intonation, the modification of the Fifth leads to a modification of the comma and obliteration of the skhisma; so that the two first tones, skhistic †f, †a, of the new upper trine, †f+†a+c♯, are *one* skhistic comma *higher*, and the third, skhistic c♯, is *one* skhistic comma *lower* than the two last tones, F, A, and the first tone D♭ of the old trine, D♭+F+A. And the tones of the new right quaternion will be in the same order, exactly *two* skhistic commas *flatter* than the old left-hand quaternion[1].

The consequence is that if we took 4 *independent* duodenes (that is, such that no tone of one is common to any tone of the other) as the duodenes of †B♭, A♯, G♭, and ‡F♯, the tones of which are contained within the dotted lines and right side of the inner oblong of Table I., the tones of the duodenes A♯ and ‡F♯ will be two commas flatter than those of †B♭ and G♭; and the tones of the *two first* quaternions of the †B♭ and A♯ duodenes will be one comma *sharper* than those of the *two last* quaternions of G♭ and ‡F♯, while the tones of the *third* quaternions of †B♭ and A♯ will be one comma *flatter* than those of the duodenes of G♭ and ‡F♯ respectively.

The result, then, is that the 48 tones will consist of four corresponding sets of 12 tones each appearing in 4 forms, differing in pitch by one skhistic comma. This will appear more clearly by the following Table, in which the value in octs of the cycle of 301 is given for 73 tones, being those in cols. I. to VI. of Table I., less those in col. I., lines *l*, *m*, *n*, and col. VI., lines *y*, *z*. The 48 of those tones contained in 4 independent duodenes are in Roman capitals, the other of the 73 tones, which are some of their skhistic synonyms, are in Roman small letters, and other synonyms are added in *Italics*; the whole are divided into groups of 4, the constituents of which differ from one another by 5 octs, or one skhistic comma.

[1] This is readily seen by expressing the tones in terms of the octs of the cycle of 301, by continually adding and subtracting 176 for the Fifths and 97 for the major Thirds, adding or subtracting 301 as often as is necessary to reduce to the same Octave. A skhistic comma is represented by 5 octs. This gives

Tones.					*Octs.*				
†d♭	†f	†a	c♯	e♯	33	130	227	23	120
†g♭	†B♭	D	F♯	a♯	158	255	51	148	245
†c♭	†E♭	G	B	‡d♯	283	79	176	273	69
†f♭	†A♭	C	E	‡g♯	107	204	0	97	194
†b♭♭	D♭	F	A	‡c♯	232	28	125	222	18
†e♭♭	g♭	b♭	‡d	‡f♯	56	153	250	46	143

Tones.	Octs.	Tones.	Octs.	Tones.	Octs.
‡C♯ ‡‡*d*♭ b♯♯	18	‡E♯ ‡‡*f* ‡‡‡*g*♭♭	115	‡G♯♯ ‡a ‡‡*b*♭♭	217
C♯ ‡d♭ †*b*♯♯	23	E♯ ‡f ‡‡*g*♭♭	120	g♯♯ A ‡*b*♭♭	222
†c♯ D♭ ††*b*♯♯	28	†e♯ F ‡*g*♭♭	125	†g♯♯ †A b♭♭	227
††*c*♯ †D♭ †††*b*♯♯	33	††*e*♯ †F g♭♭	130	††g♯♯ ††a †B♭♭	232
‡C♯♯ ‡‡*d* ‡‡*e*♭♭	41	‡F♯ ‡‡*g*♭ e♯♯	143	‡A♯ ‡‡*b*♭	240
c♯♯ ‡D ‡*e*♭♭	46	F♯ ‡*g*♭ †*e*♯♯	148	A♯ ‡*b*♭	245
†*c*♯♯ D *e*♭♭	51	†f♯ G♭ ††*e*♯♯	153	†a♯ B♭	250
††*c*♯♯ †D †*e*♭♭	56	††*f*♯ †G♭ †††*e*♯♯	158	††*a*♯ †B♭	255
‡‡D♯ ‡‡*e*♭	64	‡F♯♯ ‡‡*g* ‡‡*a*♭♭	166	a♯♯ ‡B ‡‡*c*♭	268
‡D♯ ‡*e*♭	69	f♯♯ ‡G ‡*a*♭♭	171	†*a*♯♯ B ‡*c*♭	273
d♯ E♭	74	†*f*♯♯ G *a*♭♭	176	††*a*♯♯ †b C♭	278
†d♯ †E♭	79	††*f*♯♯ †g †A♭♭	181	†††*a*♯♯ ††*b* †C♭	283
‡D♯♯ ‡e ‡‡*f*♭	92	‡G♯ ‡*a*♭	194	‡B♯ ‡‡*c* ‡‡‡*d*♭♭	291
d♯♯ E ‡*f*♭	97	G♯ a♭	199	B♯ ‡c ‡‡*d*♭♭	296
†*d*♯♯ †E f♭	102	†*g*♯ †A♭	204	†*b*♯ C ‡*d*♭♭	0
††*d*♯♯ ††*e* †F♭	107	††*g*♯ ††A♭	209	††*b*♯ †C d♭♭	5

Since, then, the duodene of C is precisely adapted for placing on our ordinary manuals, and no corresponding tones which have to be introduced within these limits will be more than two or three commas sharper or flatter than these, such corresponding tones (owing to our habits of reading musical notes into directions for using digitals) will be all fitted for being played on the same digitals. This is the most important point in the practical construction of instruments, and is for the first time pointed out in this paper.

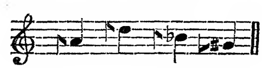
Another important result is, that if we take any 12 consecutive skhistic tones in order of Fifths, or 8 consecutive tones in that order, and 4 others separated from them by 24 or 48 Fifths, although such tones will not form a duodene, they will be 12 tones suitable for our manuals, and will therefore afford the means of temporarily supplementing other arrangements.

In skhistic intonation, then, the modulational peculiarities of just intonation are preserved; and it will be convenient in future to consider modulation as taking place by duodenes, and hence consisting of *duodenation.* We shall therefore have in just intonation both vertical and lateral duodenation to consider; but in skhistic intonation it will be seen by Table II. that *one right lateral duodenation*, as from root †B♭ to root D, is the same as *eight descending vertical duodenations*, for these would in just intonation lead to the root E♭♭, which, as shown in the last Table, is skhistically identical with D. Hence in skhistic intonation we have, so far as instruments are concerned, only to render vertical duodenation possible and easy.

And in writing music, if we note at the top of any bar the name of the duodene to which the notes to be played belong, and suppose this *duodenal* (as the mark may be called, in contradistinction to the *signature*, which will remain as before) to hold till a new one is written (according to the custom of musical signatures), we shall be able precisely to mark the pitch of every tone in just or skhistic intonation, *without introducing any change or any additional sign into the staff-notation of music*[1]. This is again an entirely new practical principle resulting from the present theory. The duodenal will direct the player to the mode of arranging the manual he has to use. It should be the duty of the composer to insert the duodenals himself; but in respect of existing compositions, which were composed for some commatic system of temperament, it will be often difficult to determine which of two adjacent vertical duodenes it would be best to use; and it will probably be necessary to introduce commatic changes, when they can be made within the limits of a single heptadecad. Also in the case of compositions in a *major* scale, which do not change into the minors of the same decad, and hence use only two quaternions of a duodene, but will necessarily and frequently modulate to the right, it is more convenient to consider the music as performed in the first and second quaternions of a duodene having for its root the Third of the major scale, because the third quaternion of that duodene contains the tones required for right lateral modulation. Thus C major will be assigned to the duodene of E, No. 19 of Table II., and F major to the duodene of A, No. 20 of Table II., &c. This makes the modulation from the major into the relative minor as simple and direct as vertical modulation, for C major passes into any form of A minor or major by descending vertically from the duodene of E to that of A. All pieces in any minor scale pass into each of the three quaternions of a duodene, and hence their duodenal will be the tonic of their decad, which gives its name to the duodene. The duodene is then prepared for playing the synonymous major of that minor scale. Such duodenals might be distinguished by an added star.

It often happens that passing tones, changing notes and *appoggiature*, are introduced which do not belong to the harmony. They are written usually after the laws of Pythagorean temperament, but their pitch is really indeterminate. For these there is no occasion to change the duodenal at all. They will then be played in the duodene of the other

[1] For theoretical and experimental purposes it may be sometimes convenient to use signs equivalent to † ‡ in the staff-notation itself. The signs for †, ††, and for ‡, ‡‡, being the tails of quavers and semiquavers, are well adapted for this purpose. The direction of the angles show ascent and descent, and the forms exist as types for every required position on the staff; thus †a, †d, †b♭, and ‡g♯, would be



harmonies by a tone of not more than two commas different, which must be considered as their proper representative in just intonation.

E. *Number of Tones required.*

The next important point is to determine how many duodenes must be provided. In Table I. the large *inner* oblong contains all the duodenes which have at least *one* tone in common with the original duodene of C. Thus the duodenes †D♭ and ‡B have respectively the tones †B♭ and A in common with the original duodene, and no others. If we proceeded further, as vertically from the †D♭ to the ††A♭ duodene, we should no longer have any connexion with the decad of the original tonic C. The confusions of modern equal temperament might lead much further; but in that case we must restore the commatic changes which equal temperament ignored, considering, for example, that when the composer modulated into tempered A♭ from tempered D♭ he really meant to make a modulation from just D♭ into just †A♭, and not from †D♭ to just ††A♭. It would probably have never been the composer's intention to proceed to such unrelated duodenes as these two last.

The limits of the *original roots* of duodenes may be taken to be the tones of the duodene of C. Practically, composers had no others in their minds. Any smaller changes of pitch were relegated to differences in the pitch of C, whence all the others were derived. If, then, we construct the limiting duodenes to the extreme tones of the duodene of C as original roots, we shall obtain all the tones in Table I., being $9 \times 13 = 117$ in number. This is the number of tones required, therefore, in just intonation.

Skhistic intonation would introduce identifications which would reduce this number to the $9 \times 8 = 72$ tones in the lines p to x in Table I., together with *three* in col. 1, lines l, m, n, and *two* in col. 9, lines y and z, that is to 77 skhistic tones in all. The last 5 are so extremely unlikely to occur, however, that we may consider these 72 skhistic tones as sufficiently representing the whole 117 of the Table. These 72 tones form 6 independent duodenes, those of ††E♭♭, D and ‡C♯♯, and of †C♭♭, B♭ and ‡A♯. It will be shown that there is really no difficulty in playing them all, with existing means, if required; but they would not be required. The tendency of musicians is not to modulate to both right and left equally in the same piece. It has been already noted that on account of the prevalence of major scales duodenation is generally to the right. The fingering for the duodenes of †B♭ and A♯ would be the same on manuals constructed on the duodenary theory, although the tones in skhistic intonation would differ by two skhistic commas. If, then, a piece in †B♭ duodenated much to the left, that is (for skhistic intonation), ascended vertically, we could play it as A♯. It would simply be necessary to write ‡B♭ as its duodenal, as that is shown to be identical with A♯ in the last Table. We should then be able to use ascending duodenation with great ease, as shown in Table II., even if columns 1 and 2 in Table I. were omitted. Hence we may begin by cancelling columns 1 and 2 of Table II. In view of the greater frequency of right lateral or descending duodenation, we need only reject column 9 to the right[1]. We have thus reduced the just tones to the $6 \times 13 = 78$ in columns I. to VI. of Table I. The skhistic identifications reduce these further to the $6 \times 8 = 48$ tones in lines p to x of the same columns, together with the three tones in col. I., l, m, n, and the two in col. VI., y, z. As these tones may, I think, be always avoided by properly choosing the original root, motives of convenience induce me to reduce the number of skhistic tones necessary to the 48 included by the dotted lines in cols. I. to VI. of Table I.

In my former paper ('Proceedings,' vol. xiii. p. 98), not having taken a sufficiently comprehensive view of the nature of modulation, I fixed the number of just tones required at 72 instead of 117, and showed that they would reduce by skhismatic substitution (for I had not then worked out the theory of skhistic temperament) to 45; and on examination it will be found that these 45 include the 48 which I have just named, with the exception of those in col. I., lines p and q, and col. VI., line x of Table I. The tones used in Mr. Liston's organ (according to the statement I was able to give in 'Proceedings,' vol. xiii. p. 417, note §), on being treated skhistically, include 44 of these 48 tones, omitting the 4 tones in col. I., lines w, x, col. V., line x, and col. VI., line x, and introduces two others found in col. 2, lines p and x, and probably only due to his system of tuning. He has thus 46 tones in all. Gen. T. Perronet Thompson's organ (see my paper in 'Proceedings,' vol. xiii. p. 102), when similarly reduced, has 38 of my 48 tones, omitting all col. I., col. II., line x, and col. VI., line r in Table I., and retaining the two tones of col. VI., lines y, z, which I do not find necessary. Mr. Poole's latest organ (Silliman's Journal for 1867, pp. 1 to 45), after rejecting his 39 natural Sevenths, which I expressly exclude, has 61 just tones, which reduce to the 36 skhistic tones in col. II., lines q to x, cols. III., IV., and V., lines p to x, and col. VI., lines p to t. These are the principal attempts at limiting the scale actually made up to this time; and hence I conclude that my reduction to 48 skhistic tones (that is, practically, unequally just tones) would embrace almost every case, though it is conceivable that some extraordinary music might make it advisable to introduce 12 more, namely, col. 2, lines p to x, and col. 9, lines p to s, making 60 tones in all, for adding which provision should be made.

[1] If in changes of key in the movements of a long piece modulation took place first much to the left and then much to the right, we might perhaps make commatic changes between the movements, without disturbing the connexion. And when changes are introduced by successions of discords, such commatic changes could not be observed at all by the listener. By the use of the duodenal, however, they will be rendered perfectly simple to the performer.

The 48 tones thus pointed out form the 32 trines, which, with their skhistic synonyms, are shown on Table II. By taking these in quaternions we obtain 29 duodenes. In Table II. the root of the duodene is written against its uppermost trine, and hence the root itself is found in the middle of the second trine below, and the whole duodene extends to the third trine below. Trines on the same line are skhistically identical, the capitals indicating the names of the 48 selected tones of cols. I. to VI., lines p to x, in Table I. In Table III. the roots of these duodenes are arranged in 4 columns, of which each tone in the same line is skhistically identical; but, proceeding from left to right, each tone is, in just intonation, one skhisma flatter than the next adjacent tone on the right.

In Table III., also, the tones in each of the duodenes are written down in the order in which they would stand on a manual; but the skhistic identities of the central column of tones in the preceding Table of Octs (p. 19) have been used to give the same names to all the tones in one column, exclusive of the prefixed † and ‡. We thus see clearly that three new tones are introduced by each new successive duodene, two falling and one rising by a skhistic comma, as respects the tones they replace. We also see that each tone prevails through 4 consecutive duodenes, and that there are 4, and only 4, varieties of †, ‡ in each column.

This completes the *theory* of the construction of instruments, because the rest is properly the work of the mechanician, and consists simply in devising a method for bringing each of these 29 duodenes under the hand of the performer, when indicated by the duodenal. It will be sufficient here to point out a few experimental instruments, to suggest some practical forms, and to show that means of playing in just intonation with fixed tones already exist.

F. *Justly or Skhistically Intoned Instruments.*

1. *Just Concertina* (exhibited when this paper was read). The C concertina described in my former paper (Proceedings, vol. xiii. p. 104) contains the portion of a heptadecad shown in the margin—that is, the duodene of E with the exception of A♯, and the duodene of A with the exception of ‡F♯. It has the whole decad of E, and the major scales of F, C, G, E. I have found it a most useful instrument in all my experiments. Using capitals for white and small letters for black studs, its 14 notes are tuned thus :—

†A		
D	F♯	
G	B	‡D♯
C	E	‡G♯
F	A	‡C♯
B♭	‡D	

C ‡c♯, D ‡d, E ‡d♯, F f♯, G ‡g♯, †A a, B b♭.

†F		
†B♭	D	
†E♭	G	B
†A♭	C	E
D♭	F	A

2. *Just Harmonium* (exhibited when this paper was read). For ordinary lecture and illustrative purposes, this is the cheapest and best instrument. It contains the portion of a heptadecad shown in the margin, and hence contains the duodene of C with the exception of F♯, the whole decad of C, and the major scales of †E♭ and †A♭. The two last show vertical modulation. Again, †E♭ major or †A♭ major to C decad shows right lateral, and, inversely, C decad to †E♭ major or †A♭ major shows left lateral modulation. The F and †F show the influence of a comma, and the difference between the just triad, D♭+F−†A♭, and the Pythagorean triad, D♭ with †F and †A♭. Also in the key of †E♭ major the difference can be shown between the minor chord, F−†A♭+C, and the chord of the added Sixth, †A♭+C | †F. It also contains the German Sixth, D♭+F−†A♭ with B (which is a close imitation of the chord of the natural Seventh), the Italian Sixth, D♭+F with B, and the French Sixth, D♭+F ... G+B. Using capitals and small letters for the black and white keys, they are arranged thus :—

C d♭ D †e♭ E F †f G †a♭ A †b♭ B.

The †F is placed on the F♯ digital, and the fingering is normal in other respects. The five tones, d♭ × †a♭ × †e♭ × †b♭ × †f, will then enable us to play all Scotch and other music containing only five tones, in perfectly just intonation, on the black digitals, and to show that such music cannot be harmonized. The practical direction for tuning is, "tune the following 7 major chords without beats, putting †F on the F♯ digital, F A C, C E G, G B D; D♭ F †A♭, †A♭ C †E♭, †E♭ G †B♭, †B♭ D †F." A four-Octave instrument was thus tuned in two hours.

3. *Hephtharmonium.* This requires two rows of vibrators and Mr. Saunders's "tilting action," already described. The vibrators are disposed thus :—

Back	†D♭	‡c♯	‡D	‡d♯	†F♭	†F	‡f♯	†G♭	‡g♯	‡A	b♭	†C♭
Front	C	d♭	D	†e♭	E	F	f♯	G	†a♭	A	†b♭	B.

This harmonium contains all the tones in a heptadecad (whence its name), and consequently illustrates every kind of modulation in the first degree, and becomes a most valuable instrument for the lecturer and teacher of singing. The front vibrators, to which correspond the digitals when the stops are all pushed in, contain the whole duodene of C, and the stops enable the player to exchange their tones for those in the six other decads. All lie on the usual digitals except †D♭ and †G♭, which are placed on the long white digitals of C and G, in place of the short black digitals next to their right, as these were wanted for ‡c♯ and ‡g♯. This makes a slight difference in the fingering of †E♭ minor and similar scales.

4. *Helmholtz's Harmonium* (Tonempfindungen, p. 496). Two sets of vibrators are tuned, the back set to the duodene of E♭ or D♯, No. 14, Tables II. and III., and the front set to duodene of ‡C♭ or B, No. 18, of the same. This instrument contains the eight trines, Nos. 14 to 21, and the five duodenes, Nos. 14 to 18, Table II. The "tilting action" produces a most useful experimental instrument, which is far easier to use than Helmholtz's own double manual instrument, because it has only one manual, and requires no alteration in ordinary fingering. For this purpose the stops may be reduced to four, each changing a trine instead of a single note.

5. *Guéroult's Harmonium* (Comptes Rendus, 1872, p. 1188). This, again, may be treated as the last by means of Mr. Saunders's "tilting action." Two sets of vibrators must be used, the back set tuned to duodene ‡D♭ or C♯, No. 16, and the front to ‡B♭♭ or A, No. 20, of Tables II. and III. M. Guéroult tuned the B♯ of duodene No. 16 as $B'\sharp = \frac{54638}{54675} B\sharp$, so as to make the combinational tone of G♯ and B'♯ the same as that of B'♯ and E♭, the other tones being tuned in just intonation from C. Omitting this as unnecessary, the instrument contains the eight trines, Nos. 16 to 23 of Table II., and the five duodenes, Nos. 16 to 20. M. Guéroult arranged the tones somewhat differently for two manuals.

6. *Duóni, Trióni, Quartóni, Quintóni, Sestóni.* The Russian horn-band which visited London some years ago, and produed great effects by each performer's playing a single tone only (and hence, probably, in just intonation), and the customs of hand-bell and church-bell ringers, who each play a single note in a melody, have suggested to me the use of *two, three, four, five,* or *six* harmoniums or pianofortes, indicated by the above names, for the purpose of playing in skhistic or unequally just intonation, by means of two, three, four, five, or six performers, among whom the tones are distributed. The *Duóni* are intended for two independent duodenes, as in the two last cases, the *Quartóni* for four such, playing the whole 48 tones, the *Sestóni* for six, in the almost impossible case of 72 tones being required. The *Trióni* supplement the *Duóni* by using 12 additional tones, forming consecutive Fifths, and hence not constituting a duodene, by which means the 36 tones of Mr. Poole's compass can be played. The *Quintóni* supplement the *Quartóni* in a similar manner; but the first 8 tones are those in col. 2, lines *p* to *x*, and the last 4 those in col. 9, lines *p* to *s* of Table II.—giving 60 tones on the whole, chosen so as to supplement without changing the arrangement of the *Quartóni.*

In each case separate harmoniums or pianos are used, with no change in existing mechanism or fingering, but only in intonation; so that the instruments could be obtained and tuned in unequally just intonation, as already described, without difficulty, at a day's notice. The music is to be marked with the proper duodenals, and the duodenes thus indicated are to be transcribed separately, and divided into parts by transverse lines, corresponding to the tones existing on the different instruments. The copyist writes out a separate part for each performer (which had better have an indication of the complete harmony annexed), in which only those notes that belong to his own instrument are written. Thus, suppose that the duodene is F♯, and the lines show what tones lie on the instrument (I., II., III.), as in the margin. Suppose that the succession of chords *e♯ g♯ b' c''♯*, *f♯ c'♯ a'♯ c''♯*, and *B ‡d'♯ f♯ b'* has to be played. The tones will be distributed as in the margin. Considerable practice would be necessary to take up the notes truly at the right moment, but *there is no longer any instrumental or digitational difficulty in playing in just intonation.*

(II.)	†E	G♯	B♯	
	†A	C♯	E♯	
(I.)	D	F♯	A♯	(III.)
	G	B	‡D♯	

(I.)	*c''♯*, *b'*	*c''♯*, *c'♯*, *f♯*	*b'*, *f'♯*, *B*
(II.)	*g♯*	—	—
(III.)	*e♯*	*a'♯*	*‡d'♯*

Leaving *Duóni* aside as sufficiently indicated in the two last cases, and of only experimental interest, and *Sestóni* as practically not required, it will be enough to explain the tuning of *Trióni*, *Quartóni*, and *Quintóni*.

Trióni. Tune the three instruments thus :—

(I.)	C	‡d♭	D	†e♭	E	†F	‡g♭	G	†a♭	†A	†b♭	B
(II.)	‡C	d♭	‡D	e♭	†E	F	g♭	‡G	a♭	A	b♭	†B
(III.)	‡B♯	‡c♯	‡C♯♯	‡d♯	‡E	E♯	‡f♯	‡F♯♯	‡g♯	‡G♯♯	a♯	‡B

Then (I.) is in the duodene of G, No. 10, and (II.) in that of E♭, No. 14 of Tables II. and III.; and this readily gives the method of tuning them. (III.) consists of 12 tones forming consecutive Fifths from E♯ to ‡E, col. 7, lines *q* to *y*, and from ‡G♯♯ to ‡B♯, col. 8, lines *q* to *t* of Table I. For (III.) begin by tuning ‡G♯ a major Third without beats to E in (I.), and then work up to E♯ and down to ‡E by Fifths, verifying with the corresponding major Thirds below in (I.) and (II.). Then tune ‡B♯ a major Third above ‡G♯, and tune up by Fifths to ‡G♯♯, verifying by the major Thirds below, which lie all in (III.). By this the three instruments are completely in tune, and give the 17 duodenes, Nos. 10 to 26, Table II., containing Mr. Poole's scale of 36 tones.

Quartóni are much simpler, because they contain the four independent duodenes,

(I.) of †B♭, No. 1,	(III.) of A♯, No. 25,
(II.) of G♭, No. 5,	(IV.) of ‡F♯, No. 29

in Table III., where the corresponding lines give the tuning of each instrument. The arrangement of tones is managed as before. Suppose, for example, we have to play the succession of chords *e g c' g'*, *f f c' a'*, *d g f' b'*, and *c g e' c'* in the duodene of C, this duodene would be written and the tones would be distributed among the four instruments as follows :—

(I.)	†B♭ D	F♯	(III.)
	†E♭ G	B	
(II.)	†A♭ C	E	(IV.)
	D♭ F	A	

(I.)	*g g'*	—	*g d*	*g*
(II.)	*c*	*c' f*	*f'*	*c c'*
(III.)	—	—	*b'*	—
(IV.)	*e*	*a'*	—	*e'*

Quintóni have five instruments, (I.) to (IV.) being tuned as in *Quartóni*, and (V.) added when by some extraordinary vagaries of modulation more than 48 tones are needed. (V.) is tuned thus, where the synonyms show the meaning of the arrangement :—

(V.)	†D♭♭	‡b♯♯	††E♭♭	†f♭♭	††F♭	†G♭♭	‡e♯♯	††A♭♭	‡f♯♯♯	††B♭♭	†c♭♭	‡A♯♯
	=††C	‡‡c♯	††D	††e♭	†††E	††F	‡‡f♯	††G	‡‡g♯	†††A	††b♭	‡‡B

The tuning of (V.) is effected thus :—Tune ††A♭♭ as a major Third below †C♭ on (I.), and then work up by Fifths to ††F♭ and down to †F♭♭, verifying by the major Thirds above in (I.) and (II.). Then tune ‡A♯♯ as a major Third above ‡F♯♯ in (III.), and work up by Fifths to ‡F♯♯♯, verifying by the major Thirds below in (III.). The notation of the tones, though inevitable, is frightful; but the tuning is very simple, and the use of the duodenal leaves the old staff-notation unchanged. It is most probable that the fifth instrument would never be wanted.

7. *Great and Small Duodenary Harmonium.* Although the mode just explained places just intonation at the immediate command of three or four performers, yet it seems necessary to suggest a mode of putting all the 17 or 29 duodenes at the command of a single performer. I suggest the following for consideration. It seems practicable, but would doubtless require much mechanical treatment from harmonium-builders before it would act properly. It will be enough to indicate the form of the great duodenary.

Take four sets of vibrators, tuned as for *Quartóni*, and placed one behind the other, each opening with a separate valve connected with a digital. Sometimes two digitals will have to be connected with the same valve. Conceive the manual as a set of 29 "steps," with $\frac{3}{4}$-inch "tread" and $\frac{1}{4}$-inch "rise," the lowest step next the performer. Each step for the length of an Octave is divided into 12 digitals corresponding to the columns in Table III. The width of the digitals to be as follows for No. 11 of Table III., in *eighths of an inch*:—

C	d♭	D	†e♭	E	F	‡g♭	G	†a♭	A	†b♭	B
5	3	5	3	5	5	3	5	3	5	3	5

The digitals corresponding to the small letters are to rise $\frac{1}{4}$ inch above the others and to be bevelled, so that they are $\frac{3}{8}$ inch wide at bottom, and $\frac{1}{4}$ inch wide at top. Each step is then a miniature finger-board in the ordinary arrangement. Whenever any note occurs in 4 consecutive steps, as shown by the cross lines in Table III., its 4 digitals are to be consolidated into one, so that, except in "steps" 1 to 3 and 26 to 29, the digitals will be practically 3 inches long. To show which digitals are consolidated, colour the low wide digitals alternately white and light red, and the high narrow digitals alternately light blue and light brown, distinctions of colour easily seen. To mark the duodene, draw a black line, $\frac{1}{4}$ inch broad, across the digital bearing the name of the duodene, and put a black circle of $\frac{1}{4}$ inch in diameter on the tonic of the major scale which it contains. The lines thus marked, together with the alternation of colour, will clearly distinguish each duodene.

The depth of this manual from front to back would be $21\frac{3}{4}$ inches, and the rise $7\frac{1}{4}$ inches; the width of an Octave from *C* to *B* is $6\frac{1}{4}$ inches, and from *C* to *c* is $6\frac{7}{8}$ inches. This last width is $7\frac{3}{8}$ inches on the piano; but as the hand would on the duodenary always have to dip between high digitals to strike Octaves of low digitals, it must be held more upright, and hence its span will be less. A manual of five Octaves and one note, *C* to c'''', will be $31\frac{7}{8}$ inches long. The number of movable digitals in each column of Table III. is 8, which open only 4 valves; this will necessitate coupling—the details resulting from Table III., which may be considered as a ground-plan of this manual[1].

[1] When this paper was read I mentioned that the 48 tones, making 29 duodenes, of Tables II. and III. could be played on Mr. Bosanquet's "generalized key-board," as exhibited to the Royal Society when his paper was read on January 30, 1873, with less difficulty in mechanism than by the plan I proposed (of which a model was exhibited), but that slightly new fingering would then be necessary; and also that the 72 tones of Table I., lines *p* to *x*, making 53 duodenes, might be played by the same arrangement on a manual not larger than that which I proposed for the 48 tones or 29 duodenes; and hence that the sole advantage of my scheme for a manual was its preservation of the present fingering, against which had to be set off the advantage that the new fingering of Mr. Bosanquet would be the same in all keys or duodenes. The intonation, however, would remain different from Mr. Bosanquet's.

Table I.

Limits of Duodenation and Number of Tones.

	1.	2.	3. (I.)	4. (II.)	5. (III.)	6. (IV.)	7. (V.)	8. (VI.)	9.	
l	†††Bbb	††Db	††F	††A	†C#	†E#	G##	B##	‡D###	*l*
m	†††Ebb	††Gb	††Bb	†D	†F#	†A#	C##	E##	‡G###	*m*
n	†††Abb	††Cb	††Eb	†G	†B	D#	F##	A##	‡C###	*n*
p	††Dbb	††Fb	††Ab	†C	†E	G#	B#	‡D##	‡F###	*p*
q	††Gbb	††Bbb	†Db	†F	†A	C#	E#	‡G##	‡B##	*q*
r	††Cbb	††Ebb	†Gb	†Bb	D	F#	A#	‡C##	‡E##	*r*
s	††Fbb	††Abb	†Cb	†Eb	G	B	‡D#	‡F##	‡A##	*s*
t	††Bbbb	†Dbb	†Fb	†Ab	C	E	‡G#	‡B#	‡‡D##	*t*
u	††Ebbb	†Gbb	†Bbb	Db	F	A	‡C#	‡E#	‡‡G##	*u*
w	††Abbb	†Cbb	†Ebb	Gb	Bb	‡D	‡F#	‡A#	‡‡C##	*w*
x	†Dbbb	†Fbb	†Abb	Cb	Eb	‡G	‡B	‡‡D#	‡‡F##	*x*
y	†Gbbb	†Bbbb	Dbb	Fb	Ab	‡C	‡E	‡‡G#	‡‡B#	*y*
z	†Cbbb	†Ebbb	Gbb	Bbb	‡Db	‡F	‡A	‡‡C#	‡‡E#	*z*

The tones in the *small* central oblong form the duodene of which C is the root.

The tones in the *large* central oblong form all the duodenes which have at least one tone in common with the central duodene of C, forming the limits of radical duodenation from C.

The complete Table contains all the duodenes which have at least one tone in common with duodenes whose roots are tones in the duodene of C, forming the limits of general radical duodenation.

The 48 tones in columns I. to VI., between the dotted lines, are those considered sufficient for instruments with fixed tones in skhistic or unequally just intonation.

Table II.

List of Trines and Duodenes in order of Fifths.

The Capitals point out the 48 tones in Table I.; the small letters are synonyms.

Name.	No.	Trine.			Name.	No.	Trine.			Name.	No.	Trine.		
†Bb	1	††Ab	†C	†E	††A#	1	††g#	††b#	†d##					
†Eb	2	†Db	†F	†A	†D#	2	††c#	††e#	†g##					
†Ab	3	†Gb	†Bb	D	†G#	3	††f#	††a#	†c##					
Db	4	†Cb	†Eb	G	†C#	4	††b	†d#	†f##					
Gb	5	†Fb	†Ab	C	†F#	5	††c	†g#	†b#					
Cb	6	†Bbb	Db	F	†B	6	††a	†c#	†e#					
Fb	7	†Ebb	Gb	Bb	†E	7	†d	†f#	†a#					
Bbb	8	†Abb	Cb	Eb	†A	8	†g	†b	d#					
Ebb	9	dbb	fb	ab	D	9	†C	†E	G#	†C##	9	††b#	†d##	†f###
Abb	10	gbb	bbb	‡db	G	10	†F	†A	C#	†F##	10	††e#	†g##	†b##
‡Dbb	11	cbb	ebb	‡gb	C	11	†Bb	D	F#	†B#	11	††a#	†c##	†e##
‡Gbb	12	fbb	abb	‡cb	F	12	†Eb	G	B	†E#	12	†d#	†f##	†a##
‡Cbb	13	bbbb	‡dbb	‡fb	Bb	13	†Ab	C	E	†A#	13	†g#	†b#	d##
	14	ebbb	‡gbb	‡bbb	Eb	14	Db	F	A	D#	14	†c#	†e#	g##
	15	abbb	‡cbb	‡ebb	Ab	15	Gb	Bb	‡D	G#	15	†f#	†a#	c##
	16	‡dbbb	‡fbb	‡abb	‡Db	16	Cb	Eb	‡G	C#	16	†b	d#	f##
†E##	17	†d##	†f###	†a###	‡Gb	17	fb	ab	‡c	F#	17	†E	G#	B#
†A##	18	†g##	†b##	d###	‡Cb	18	bbb	‡db	‡f	B	18	†A	C#	E#
D##	19	†c##	†e##	g###	‡Fb	19	ebb	‡gb	‡bb	E	19	D	F#	A#
G##	20	†f##	†a##	c###	‡Bbb	20	abb	‡cb	‡eb	A	20	G	B	‡D#
C##	21	†b#	d##	f###	‡Ebb	21	‡dbb	‡fb	‡ab	‡D	21	C	E	‡G#
F##	22	†e#	g##	b##		22	‡gbb	‡bbb	‡‡db	‡G	22	F	A	‡C#
B#	23	†a#	c##	e##		23	‡cbb	‡ebb	‡‡gb	‡C	23	Bb	‡D	‡F#
E#	24	d#	f##	a##		24	‡fbb	‡abb	‡‡cb	‡F	24	Eb	‡G	‡B
A#	25	G#	B#	‡D##						‡Bb	25	ab	‡c	‡e
‡D#	26	C#	E#	‡G##						‡Eb	26	‡db	‡f	‡a
‡G#	27	F#	A#	‡C##						‡Ab	27	‡gb	‡bb	‡‡d
‡C#	28	B	‡D#	‡F##						‡‡Db	28	‡cb	‡eb	‡‡g
‡F#	29	E	‡G#	‡B#						‡‡Gb	29	‡fb	‡ab	‡‡c
	30	A	‡C#	‡E#							30	‡bbb	‡‡db	‡‡f
	31	‡D	‡F#	‡A#							31	‡ebb	‡‡gb	‡‡bb
	32	‡G	B	‡‡D#							32	‡abb	‡‡cb	‡‡eb

Table III.

Manuals for Duodenary Instruments.

The Capital letters indicate broad and low, small letters narrow and high, digitals.

Names of Duodenes with Synonyms.	No.	Digitals, containing the tones of the Duodenes displayed horizontally.											
		1	2	3	4	5	6	7	8	9	10	11	12
†Bb ††a#	1	†C	†db	D	†eb	†E	†F	†gb	G	††ab	†A	†bb	††B
†Eb †d#	2	C	†db	D	†eb	††E	†F	†gb	G	†ab	†A	†bb	††B
†Ab †g#	3	C	db	D	†eb	††E	F	†gb	G	†ab	††A	†bb	††B
Db †c#	4	C	db	†D	†eb	††E	F	gb	G	†ab	††A	bb	††B
Gb †f#	5	C	db	†D	eb	††E	F	gb	†G	†ab	††A	bb	†B
Cb †b	6	†C	db	†D	eb	†E	F	gb	†G	ab	††A	bb	†B
Fb †e	7	†C	‡db	†D	eb	†E	†F	gb	†G	ab	†A	bb	†B
Bbb †a	8	†C	‡db	D	eb	†E	†F	‡gb	†G	ab	†A	†bb	†B
ebb D †c##	9	†C	‡db	D	†eb	†E	†F	‡gb	G	ab	†A	†bb	B
abb G †f##	10	C	‡db	D	†eb	E	†F	‡gb	G	†ab	†A	†bb	B
‡dbb C †b#	11	C	db	D	†eb	E	F	‡gb	G	†ab	A	†bb	B
‡gbb F †e#	12	C	db	‡D	†eb	E	F	gb	G	†ab	A	bb	B
‡cbb Bb †a#	13	C	db	‡D	eb	E	F	gb	‡G	†ab	A	bb	†B
Eb d#	14	‡C	db	‡D	eb	†E	F	gb	‡G	ab	A	bb	†B
Ab g#	15	‡C	‡db	‡D	eb	†E	‡F	gb	‡G	ab	†A	bb	†B
‡Db c#	16	‡C	‡db	D	eb	†E	‡F	‡gb	‡G	ab	†A	‡bb	†B
‡gb F# †e##	17	‡C	‡db	D	‡eb	†E	‡F	‡gb	G	ab	†A	‡bb	B
‡cb B †a##	18	C	‡db	D	‡eb	E	‡F	‡gb	G	‡ab	†A	‡bb	B
‡fb E d##	19	C	‡‡db	D	‡eb	E	F	‡gb	G	‡ab	A	‡bb	B
‡bbb A g##	20	C	‡‡db	‡D	‡eb	E	F	‡‡gb	G	‡ab	A	bb	B
‡ebb ‡D c##	21	C	‡‡db	‡D	eb	E	F	‡‡gb	‡G	‡ab	A	bb	‡B
‡G f##	22	‡C	‡‡db	‡D	eb	‡E	F	‡‡gb	‡G	ab	A	bb	‡B
‡C b#	23	‡C	‡db	‡D	eb	‡E	‡F	‡‡gb	‡G	ab	‡A	bb	‡B
‡F e#	24	‡C	‡db	‡‡D	eb	‡E	‡F	‡gb	‡G	ab	‡A	‡bb	‡B
‡bb A#	25	‡C	‡db	‡‡D	‡eb	‡E	‡F	‡gb	‡‡G	ab	‡A	‡bb	B
‡eb ‡D#	26	‡‡C	‡db	‡‡D	‡eb	E	‡F	‡gb	‡‡G	‡ab	‡A	‡bb	B
‡‡ab ‡G#	27	‡‡C	‡‡db	‡‡D	‡eb	E	‡‡F	‡gb	‡‡G	‡ab	A	‡bb	B
‡‡db ‡C#	28	‡‡C	‡‡db	‡D	‡eb	E	‡‡F	‡‡gb	‡‡G	‡ab	A	‡‡bb	B
‡‡gb ‡F#	29	‡‡C	‡‡db	‡D	‡‡eb	E	‡‡F	‡‡gb	‡G	‡ab	A	‡‡bb	‡B